

# ON THE HARMONIC OSCILLATOR GROUP

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**ABSTRACT.** We discuss the maximum kinematical invariance group of the quantum harmonic oscillator from a viewpoint of the Ermakov-type system. A six parameter family of the square integrable oscillator wave functions, which seems cannot be obtained by the standard separation of variables, is presented as an example. The invariance group of the generalized driven harmonic oscillator is shown to be isomorphic to the corresponding Schrödinger group of the free particle.

Quantum systems with quadratic Hamiltonians are called the generalized harmonic oscillators (see, for example, [9], [12], [14], [28], [46], [47], [48] and the references therein). These systems have attracted substantial attention over the years because of their great importance in many advanced quantum problems. Examples are coherent states and uncertainty relations, Berry's phase, quantization of mechanical systems and Hamiltonian cosmology. More applications include, but are not limited to charged particle traps and motion in uniform magnetic fields, molecular spectroscopy and polyatomic molecules in varying external fields, crystals through which an electron is passing and exciting the oscillator modes, and other mode interactions with external fields. Quadratic Hamiltonians have particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators [14].

The purpose of this paper is to give a simple derivation of the maximum kinematical invariance groups of the free particle and harmonic oscillator, which were introduced in Refs. [3], [4], [17], [31] and [32] (see also [5] and the references therein), from a unified approach to generalized harmonic oscillators (see, for example, [7], [9], [23] and the references therein). Relations with the corresponding Riccati and Ermakov-type systems, which seem to be missing in the available literature, are emphasized.

## 1. TRANSFORMING NONAUTONOMOUS SCHRÖDINGER EQUATION INTO AUTONOMOUS FORM

Quantum systems described by the one-dimensional time-dependent Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi, \quad (1.1)$$

where the variable Hamiltonian  $H = Q(p, x)$  is an arbitrary quadratic of two operators  $p = -i\partial/\partial x$  and  $x$ , namely,

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x \quad (1.2)$$

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( $a, b, c, d, f$  and  $g$  are suitable real-valued functions of time only), are known as the *generalized (driven) harmonic oscillators*. Some examples, a general approach and known elementary solutions can be found in Refs. [7], [8], [9], [10], [12], [14], [26], [37], [46] and [47].

In this paper, we shall use the following result established in [23].

**Lemma 1.** *The substitution*

$$\psi = \frac{e^{i(\alpha(t)x^2 + \delta(t)x + \kappa(t))}}{\sqrt{\mu(t)}} \chi(\xi, \tau), \quad \xi = \beta(t)x + \varepsilon(t), \quad \tau = -\gamma(t) \quad (1.3)$$

*transforms the non-autonomous and inhomogeneous Schrödinger equation (1.2) into the autonomous form*

$$i\chi_\tau = -\chi_{\xi\xi} + c_0\xi^2\chi \quad (c_0 = 0, 1) \quad (1.4)$$

*provided that*

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0a\beta^4, \quad (1.5)$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \quad (1.6)$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0 \quad (1.7)$$

*and*

$$\frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2g\alpha + 2c_0a\beta^3\varepsilon, \quad (1.8)$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta)\beta, \quad (1.9)$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0a\beta^2\varepsilon^2. \quad (1.10)$$

*Here*

$$\alpha = \frac{1}{4a} \frac{\mu'}{\mu} - \frac{d}{2a}. \quad (1.11)$$

(We have changed  $\tau \rightarrow -\tau$  in this paper for convenience. A **Mathematica** proof of Lemma 1 is given by Christoph Koutschan [20]; see also [21]. A special case of the substitution (1.3) has been named the Quantum Arnold Transformation in the recent publications [2], [16] and [27]; see also [23], [48] and references therein for the earlier works.)

The substitution (1.11) reduces the inhomogeneous equation (1.5) to the second order ordinary differential equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = c_0(2a)^2\beta^4\mu, \quad (1.12)$$

that has the familiar time-varying coefficients

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right). \quad (1.13)$$

When  $c_0 = 0$ , equation (1.5) is called the *Riccati nonlinear differential equation* [43], [45] and the system (1.5)–(1.10) shall be referred to as a *Riccati-type system*. (Similar terminology is used in [39], [40] for the corresponding parabolic equation.) If  $c_0 = 1$ , equation (1.12) can be reduced to a generalized version of the *Ermakov nonlinear differential equation* (see, for example, [9], [13], [25], [41] and the references therein regarding Ermakov's equation) and we shall refer to the corresponding

system (1.5)–(1.10) with  $c_0 \neq 0$  as an *Ermakov-type system*. Throughout this paper, we use the notations from Ref. [23] where a more detailed bibliography on the quadratic systems can be found. We have to remind to the reader how to solve the systems (1.5)–(1.10) (in quadratures) in order to make our presentation as self-contained as possible.

The time-dependent coefficients  $\alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0, \kappa_0$  that satisfy the Riccati-type system (1.5)–(1.10) are given as follows [7], [37], [41]:

$$\alpha_0(t) = \frac{1}{4a(t)} \frac{\mu'_0(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \quad (1.14)$$

$$\beta_0(t) = -\frac{\lambda(t)}{\mu_0(t)}, \quad \lambda(t) = \exp\left(-\int_0^t (c(s) - 2d(s)) ds\right), \quad (1.15)$$

$$\gamma_0(t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{d(0)}{2a(0)} \quad (1.16)$$

and

$$\delta_0(t) = \frac{\lambda(t)}{\mu_0(t)} \int_0^t \left[ \left( f(s) - \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu'_0(s) \right] \frac{ds}{\lambda(s)}, \quad (1.17)$$

$$\begin{aligned} \varepsilon_0(t) = & -\frac{2a(t)\lambda(t)}{\mu'_0(t)} \delta_0(t) + 8 \int_0^t \frac{a(s)\sigma(s)\lambda(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s)) ds \\ & + 2 \int_0^t \frac{a(s)\lambda(s)}{\mu'_0(s)} \left( f(s) - \frac{d(s)}{a(s)} g(s) \right) ds, \end{aligned} \quad (1.18)$$

$$\begin{aligned} \kappa_0(t) = & \frac{a(t)\mu_0(t)}{\mu'_0(t)} \delta_0^2(t) - 4 \int_0^t \frac{a(s)\sigma(s)}{(\mu'_0(s))^2} (\mu_0(s)\delta_0(s))^2 ds \\ & - 2 \int_0^t \frac{a(s)}{\mu'_0(s)} (\mu_0(s)\delta_0(s)) \left( f(s) - \frac{d(s)}{a(s)} g(s) \right) ds \end{aligned} \quad (1.19)$$

( $\delta_0(0) = -\varepsilon_0(0) = g(0)/(2a(0))$  and  $\kappa_0(0) = 0$ ) provided that  $\mu_0$  and  $\mu_1$  are standard solutions of equation (1.12) with  $c_0 = 0$  corresponding to the initial conditions  $\mu_0(0) = 0, \mu'_0(0) = 2a(0) \neq 0$  and  $\mu_1(0) \neq 0, \mu'_1(0) = 0$ . (Proofs of these facts are outlined in Refs. [7], [11] and [37]. Extensions to the nonlinear Schrödinger equations are discussed in Refs. [1], [6], [38] and [42].)

Solution of the Riccati-type system in terms of a nonlinear superposition principle is considered in [37].

**Lemma 2.** *The solution of the Riccati-type system (1.5)–(1.10) is given by*

$$\mu(t) = 2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t)), \quad (1.20)$$

$$\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \quad (1.21)$$

$$\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))} = \frac{\beta(0)\mu(0)}{\mu(t)}\lambda(t), \quad (1.22)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)}{4(\alpha(0) + \gamma_0(t))} \quad (1.23)$$

and

$$\delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (1.24)$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (1.25)$$

$$\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))} \quad (1.26)$$

in terms of the fundamental solution (1.14)–(1.19) subject to the arbitrary (finite) initial data  $\mu(0)$ ,  $\alpha(0)$ ,  $\beta(0) \neq 0$ ,  $\gamma(0)$ ,  $\delta(0)$ ,  $\varepsilon(0)$ ,  $\kappa(0)$ .

The following extension allows to solve the Ermakov-type system [23].

**Lemma 3.** *The solution of the Ermakov-type system (1.5)–(1.10) when  $c_0 = 1 (\neq 0)$  is given by*

$$\mu = \mu(0) \mu_0 \sqrt{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2}, \quad (1.27)$$

$$\alpha = \alpha_0 - \beta_0^2 \frac{\alpha(0) + \gamma_0}{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2}, \quad (1.28)$$

$$\beta = -\frac{\beta(0)\beta_0}{\sqrt{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2}} = \frac{\beta(0)\mu(0)}{\mu(t)} \lambda(t), \quad (1.29)$$

$$\gamma = \gamma(0) - \frac{1}{2} \arctan \frac{\beta^2(0)}{2(\alpha(0) + \gamma_0)}, \quad \alpha(0) > 0 \quad (1.30)$$

and

$$\delta = \delta_0 - \beta_0 \frac{\varepsilon(0)\beta^3(0) + 2(\alpha(0) + \gamma_0)(\delta(0) + \varepsilon_0)}{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2}, \quad (1.31)$$

$$\varepsilon = \frac{2\varepsilon(0)(\alpha(0) + \gamma_0) - \beta(0)(\delta(0) + \varepsilon_0)}{\sqrt{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2}}, \quad (1.32)$$

$$\begin{aligned} \kappa = & \kappa(0) + \kappa_0 - \varepsilon(0)\beta^3(0) \frac{\delta(0) + \varepsilon_0}{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2} \\ & + (\alpha(0) + \gamma_0) \frac{\varepsilon^2(0)\beta^2(0) - (\delta(0) + \varepsilon_0)^2}{\beta^4(0) + 4(\alpha(0) + \gamma_0)^2} \end{aligned} \quad (1.33)$$

in terms of the fundamental solution (1.14)–(1.19) subject to the arbitrary initial data  $\mu(0)$ ,  $\alpha(0)$ ,  $\beta(0) \neq 0$ ,  $\gamma(0)$ ,  $\delta(0)$ ,  $\varepsilon(0)$ ,  $\kappa(0)$ .

Using standard oscillator wave functions for equation (1.4) when  $c_0 = 1$  results in

$$\psi_n(x, t) = \frac{e^{i(\alpha x^2 + \delta x + \kappa) + i(2n+1)\gamma}}{\sqrt{2^n n! \mu \sqrt{\pi}}} e^{-(\beta x + \varepsilon)^2/2} H_n(\beta x + \varepsilon), \quad (1.34)$$

where  $H_n(x)$  are the Hermite polynomials [33] and the solution of the Ermakov-type system (1.5)–(1.10) is available [23]. They are also eigenfunctions,

$$E(t) \psi_n(x, t) = \lambda(t) \left( n + \frac{1}{2} \right) \psi_n(x, t), \quad (1.35)$$

of the corresponding dynamic invariant [36]:

$$E(t) = \frac{\lambda(t)}{2} \left[ \frac{(p - 2\alpha x - \delta)^2}{\beta^2} + (\beta x + \varepsilon)^2 \right], \quad \frac{d}{dt} \langle E \rangle = 0. \quad (1.36)$$

The Green function of generalized harmonic oscillators,

$$G(x, y, t) = \frac{1}{\sqrt{2\pi i \mu_0(t)}} \exp \left[ i \left( \alpha_0(t) x^2 + \beta_0(t) xy + \gamma_0(t) y^2 + \delta_0(t) x + \varepsilon_0(t) y + \kappa_0(t) \right) \right], \quad (1.37)$$

has been constructed in Ref. [7]. Evaluation of the Berry phase is discussed in Ref. [36].

In the reminder of the paper, we apply these general results to the maximum kinematical invariance groups of the free Schrödinger equation and of the harmonic oscillator [31], [32] (see also [5] and the references therein). In addition, our approach allows to describe the maximal kinematical invariance group of the generalized harmonic oscillators.

## 2. SPECIAL CASES

In this section, we show that the maximal kinematical invariance groups of the free Schrödinger equation [31] and the harmonic oscillator [32] (and their isomorphism) can be obtain as special cases of the transformation (1.3).

**2.1. Transformation from the free particle to the free particle.** In the simplest case  $a = 1$ ,  $b = c = d = f = g = c_0 = 0$ , one finds  $\mu_0 = 2t$ ,  $\mu_1 = 1$  and  $\alpha_0 = -\beta_0/2 = \gamma_0 = 1/(4t)$ ,  $\delta_0 = \varepsilon_0 = \kappa_0 = 0$ . The general solution of the corresponding Riccati-type system is given by

$$\mu(t) = \mu(0) (1 + 4\alpha(0)t), \quad (2.1)$$

$$\alpha(t) = \frac{\alpha(0)}{1 + 4\alpha(0)t}, \quad \beta(t) = \frac{\beta(0)}{1 + 4\alpha(0)t}, \quad (2.2)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0)t}{1 + 4\alpha(0)t}, \quad \delta(t) = \frac{\delta(0)}{1 + 4\alpha(0)t}, \quad (2.3)$$

$$\varepsilon(t) = \varepsilon(0) - \frac{2\beta(0)\delta(0)t}{1 + 4\alpha(0)t}, \quad \kappa(t) = \kappa(0) - \frac{\delta^2(0)t}{1 + 4\alpha(0)t}. \quad (2.4)$$

The Ansatz (1.3) together with these formulas determine the Schrödinger group, namely, the maximum (known) kinematical invariance group of the free Schrödinger equation, as follows [31]:

$$\begin{aligned} \psi(x, t) = & \frac{1}{\sqrt{\mu(0)(1 + 4\alpha(0)t)}} \exp \left[ i \left( \frac{\alpha(0)x^2 + \delta(0)x - \delta^2(0)t}{1 + 4\alpha(0)t} + \kappa(0) \right) \right] \\ & \times \chi \left( \frac{\beta(0)x - 2\beta(0)\delta(0)t}{1 + 4\alpha(0)t} + \varepsilon(0), \frac{\beta^2(0)t}{1 + 4\alpha(0)t} - \gamma(0) \right). \end{aligned} \quad (2.5)$$

We have established a connection of the Schrödinger group with the Riccati-type system (see also [5], [15], [19], [30], [32], [44] for the Lie group approach; subgroups of the Schrödinger group and their invariants are discussed in [5]).

The subgroups include the familiar *Galilei transformations*:

$$\psi(x, t) = \exp \left[ i \left( \frac{V}{2}x - \frac{V^2}{4}t \right) \right] \chi(x - Vt + x_0, t - t_0), \quad (2.6)$$

when  $\alpha(0) = \kappa(0) = 0$ ,  $\beta(0) = \mu(0) = 1$ ,  $\gamma(0) = t_0$ ,  $\varepsilon(0) = x_0$  and  $\delta(0) = V/2$ ; supplemented by *dilatations*:

$$\psi(x, t) = \chi(lx, l^2t) \quad (2.7)$$

with  $\alpha(0) = \gamma(0) = \delta(0) = \varepsilon(0) = \kappa(0) = 0$ ,  $\mu(0) = 1$  and  $\beta(0) = l$ ; and *expansions*:

$$\begin{aligned} \psi(x, t) &= \frac{1}{\sqrt{1+mt}} \exp \left( i \frac{mx^2}{4(1+mt)} \right) \chi \left( \frac{x}{1+mt}, \frac{t}{1+mt} \right) \\ &(\mu(0) = 1 (\neq 0), \quad \mu'(0) = m), \end{aligned} \quad (2.8)$$

$$\begin{aligned} \psi(x, t) &= \frac{1}{\sqrt{2t}} \exp \left( i \frac{x^2}{4t} \right) \chi \left( -\frac{x}{2t}, -\frac{1}{4t} \right) \\ &(\mu(0) = 0, \quad \mu'(0) = 2 (\neq 0)) \end{aligned} \quad (2.9)$$

with  $\beta(0) = 1$ ,  $\delta(0) = \varepsilon(0) = \kappa(0) = 0$ . (The symmetry group of the corresponding diffusion equation is discussed in [19], [30], [35] and [40].)

Here, we derive the Schrödinger group for the free particle as a very special case of Lemma 1. In turn, application of (2.5) in (1.3) produces a composition of these two transformations.

**2.2. Transformations from the harmonic oscillator to the free particle.** If  $a = b = 1$ ,  $c = d = f = g = c_0 = 0$ , the corresponding characteristic equation,  $\mu'' + 4\mu = 0$ , has two standard solutions  $\mu_0 = \sin 2t$ ,  $\mu_1 = \cos 2t$  and

$$\alpha_0 = \gamma_0 = \frac{\cos 2t}{2 \sin 2t}, \quad \beta_0 = -\frac{1}{\sin 2t}, \quad \delta_0 = \varepsilon_0 = \kappa_0 = 0, \quad (2.10)$$

$$\mu = \mu(0) (2\alpha(0) \sin 2t + \cos 2t). \quad (2.11)$$

The general solution of the corresponding Riccati-type system takes the form

$$\alpha(t) = \frac{2\alpha(0) \cos 2t - \sin 2t}{2(2\alpha(0) \sin 2t + \cos 2t)}, \quad (2.12)$$

$$\beta(t) = \frac{\beta(0)}{2\alpha(0) \sin 2t + \cos 2t}, \quad (2.13)$$

$$\gamma(t) = \gamma(0) - \frac{\beta^2(0) \sin 2t}{2(2\alpha(0) \sin 2t + \cos 2t)}, \quad (2.14)$$

$$\delta(t) = \frac{\delta(0)}{2\alpha(0) \sin 2t + \cos 2t}, \quad (2.15)$$

$$\varepsilon(t) = \varepsilon(0) - \frac{\beta(0) \delta(0) \sin 2t}{2\alpha(0) \sin 2t + \cos 2t}, \quad (2.16)$$

$$\kappa(t) = \kappa(0) - \frac{\delta^2(0) \sin 2t}{2(2\alpha(0) \sin 2t + \cos 2t)} \quad (2.17)$$

subject to arbitrary given initial data. Letting  $\mu(0) = \beta(0) = 1$  and  $\alpha(0) = \gamma(0) = \delta(0) = \varepsilon(0) = \kappa(0) = 0$ , we arrive at the simple substitution [32]:

$$\psi(x, t) = \frac{e^{-(i/2)x^2 \tan 2t}}{\sqrt{\cos 2t}} \chi\left(\frac{x}{\cos 2t}, \frac{\tan 2t}{2}\right) \quad (2.18)$$

(see also [15], [16], [30], [42]).

**2.3. Transformation from the free particle to the harmonic oscillator.** In the simplest case  $a = c_0 = 1$ ,  $b = c = d = f = g = 0$ , the general solution of the corresponding Ermakov-type system is given by

$$\mu(t) = \mu(0) \sqrt{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2}, \quad (2.19)$$

$$\alpha(t) = \frac{\beta^4(0)t + \alpha(0)(4\alpha(0)t + 1)}{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2}, \quad (2.20)$$

$$\beta(t) = \frac{\beta(0)}{\sqrt{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2}}, \quad (2.21)$$

$$\gamma(t) = \gamma(0) - \frac{1}{2} \arctan \frac{2\beta^2(0)t}{4\alpha(0)t + 1}, \quad (2.22)$$

$$\delta(t) = \frac{2\varepsilon(0)\beta^3(0)t + \delta(0)(4\alpha(0)t + 1)}{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2}, \quad (2.23)$$

$$\varepsilon(t) = \frac{\varepsilon(0)(4\alpha(0)t + 1) - 2\beta(0)\delta(0)t}{\sqrt{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2}}, \quad (2.24)$$

$$\begin{aligned} \kappa(t) = \kappa(0) + t \frac{(4\alpha(0)t + 1)(\varepsilon^2(0)\beta^2(0) - \delta^2(0))}{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2} \\ - t^2 \frac{4\varepsilon(0)\delta(0)\beta^3(0)}{4\beta^4(0)t^2 + (4\alpha(0)t + 1)^2}. \end{aligned} \quad (2.25)$$

When  $\mu(0) = \beta(0) = 1$ ,  $\alpha(0) = \gamma(0) = \delta(0) = \varepsilon(0) = \kappa(0) = 0$ , we get the known transformation [16], [17], [18], [27], [30], [32]:

$$\psi(x, t) = \frac{1}{(4t^2 + 1)^{1/4}} \exp\left(i \frac{tx^2}{4t^2 + 1}\right) \chi\left(\frac{x}{\sqrt{4t^2 + 1}}, \frac{1}{2} \arctan 2t\right). \quad (2.26)$$

One can easily verify that transformations (2.18) and (2.26) are (local) inverses of each other.

Equations (1.34) and (2.19)–(2.25) provide a six parameter family of the “harmonic oscillator states” for the free particle (see also [16] and the references therein).

**2.4. Transformation from the harmonic oscillator to the harmonic oscillator.** We consider the case  $a = b = c_0 = 1$ ,  $c = d = f = g = 0$ . The general solution of the corresponding Ermakov-type system is given by

$$\mu(t) = \mu(0) \sqrt{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2}, \quad (2.27)$$

$$\alpha(t) = \frac{\alpha(0) \cos 4t + \sin 4t (\beta^4(0) + 4\alpha^2(0) - 1)/4}{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2}, \quad (2.28)$$

$$\beta(t) = \frac{\beta(0)}{\sqrt{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2}}, \quad (2.29)$$

$$\gamma(t) = \gamma(0) - \frac{1}{2} \arctan \frac{\beta^2(0) \sin 2t}{2\alpha(0) \sin 2t + \cos 2t}, \quad (2.30)$$

$$\delta(t) = \frac{\delta(0) (2\alpha(0) \sin 2t + \cos 2t) + \varepsilon(0) \beta^3(0) \sin 2t}{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2}, \quad (2.31)$$

$$\varepsilon(t) = \frac{\varepsilon(0) (2\alpha(0) \sin 2t + \cos 2t) - \beta(0) \delta(0) \sin 2t}{\sqrt{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2}}, \quad (2.32)$$

$$\begin{aligned} \kappa(t) = & \kappa(0) + \sin^2 2t \frac{\varepsilon(0) \beta^2(0) (\alpha(0) \varepsilon(0) - \beta(0) \delta(0)) - \alpha(0) \delta^2(0)}{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2} \\ & + \frac{1}{4} \sin 4t \frac{\varepsilon^2(0) \beta^2(0) - \delta^2(0)}{\beta^4(0) \sin^2 2t + (2\alpha(0) \sin 2t + \cos 2t)^2}. \end{aligned} \quad (2.33)$$

(A direct substitution with the help of **Mathematica** verifies that these expressions indeed satisfy the Ermakov-type system.)

The Ansatz (1.3) together with the formulas (2.27)–(2.33) explicitly determine the harmonic oscillator group introduced in Ref. [32]. The established connection with the corresponding Ermakov-type system allows us to bypass the traditional Lie algebra approach [3], [5], [30], [32]. Combination of the transformations (2.5) and (2.9), (2.18) and (2.26) implies that the harmonic oscillator group is isomorphic to the Schrödinger group of the free particle (see (3.1) below) [32]. An analog of the similarity transformation (2.9) takes the form  $\psi(x, t) = \chi(-x, t - \pi/4)$  and further identification of the corresponding subgroups and the identity element is similar to the case of the free particle. We have derived the harmonic oscillator group as a special case of Lemma 1 and one can consider a composition of these transformations.

Use of the explicit solution (2.27)–(2.33) in the wave function (1.34) results in a six parameter (square integrable) family of the “dynamic quantum oscillator states”, which seems cannot be obtained by the standard separation of variables (the case  $\beta(0) = 1$  and  $\alpha(0) = \gamma(0) = \delta(0) = \varepsilon(0) = \kappa(0) = 0$  corresponds to the textbook oscillator solution [22], [29]). The corresponding quadratic dynamic invariant (1.36) and the creation and annihilation operators are found (in general) in Ref. [36]. A generalization of the coherent states is discussed in Ref. [24].

### 3. THE MAXIMAL KINEMATICAL INVARIANCE GROUP OF THE GENERALIZED DRIVEN HARMONIC OSCILLATORS

The transformation (1.3) from Lemma 1 admits an inversion when the coefficient  $a(t)$  does not change its sign (see (1.7) for the monotonicity and local time inversion). As a result, the invariance group of the generalized driven harmonic oscillator is isomorphic to the Schrödinger group of the free particle,

$$T = S^{-1} T_0 S, \quad (3.1)$$



thus extending the result of [32] to the corresponding nonautonomous systems (in the classical case, see (2.18) and (2.26) for possible operators  $S$  and  $S^{-1}$  and the operator  $T_0$  is defined by (2.5) and (2.9)). The structure of the Schrödinger group of operators  $T_0$  in two-dimensional space-time as a semidirect product of  $SL(2, \mathbb{R})$  and Weyl  $W(1)$  groups is discussed, for example, in Refs. [5], [19] and [30]. Further details are left to the reader.

#### 4. A CONCLUSION

Our analysis of the maximal kinematical invariance group of the quantum harmonic oscillator provides a six parameter family of solutions, namely (1.34) and (2.27)–(2.33), for the arbitrary initial data of the corresponding Ermakov-type system. These “hidden parameters” disappear after evaluation of matrix elements and cannot be observed from the spectrum. How to distinguish between these “new dynamic” and the “standard” harmonic oscillator states is thus an open problem. (The probability density  $|\psi|^2$  of the solution (1.34) is moving with time even for the oscillator “dynamic ground state” – a simple `Mathematica` animation reveals such oscillations. This effect, quite possibly, can be observed experimentally.) This ambiguity, which is due to the nontrivial oscillator maximal kinematical invariance group, then goes further into the coherent states, evaluation of Berry’s phase and dynamic invariants through the established connection with solutions of the Ermakov-type system. All of that puts considerations of this paper into a somewhat broader mathematical and physical context.

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